Full wave-field reflection coefficient inversion

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This paper develops a Bayesian inversion for recovering multilayer geoacoustic (velocity, density, attenuation) profiles from a full wave-field (spherical-wave) seabed reflection response. The reflection data originate from acoustic time series windowed for a single bottom interaction, which are processed to yield reflection coefficient data as a function of frequency and angle. Replica data for inversion are computed using a wave number-integration model to calculate the full complex acoustic pressure field, which is processed to produce a commensurate seabed response function. To address the high computational cost of calculating short range acoustic fields, the inversion algorithms are parallelized and frequency averaging is replaced by range averaging in the forward model. The posterior probability density is interpreted in terms of optimal parameter estimates, marginal distributions, and credibility intervals. Inversion results for the full wave-field seabed response are compared to those obtained using plane-wave reflection coefficients. A realistic synthetic study indicates that the plane-wave assumption can fail, producing erroneous results with misleading uncertainty bounds, whereas excellent results are obtained with the full-wave reflection inversion. © 2007 Acoustical Society of America. [DOI: 10.1121/1.2793609]

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I. INTRODUCTION

Geoacoustic inversion provides estimates of physical seabed parameters without the need of direct measurements (e.g., coring). Knowledge of these parameters is of interest to a wide range of applications in underwater acoustics, including acoustic propagation modeling and source localization. Single-bounce reflectivity measurements have been developed to resolve local sediment structure using plane wave approximations.1–3 Holland and Osler4 developed an experimental technique to measure high-resolution seabed reflectivity in shallow water, using a moving impulsive source (e.g., seismic boomer). For impulsive sources, Stickler5 and Schmidt and Jensen6 showed that the plane wave approximation is often violated, and suggested taking spherical wave effects into account when calculating reflection coefficients. Holland and Osler4 analyzed their data including spherical wave effects in the ray theory approximation, i.e., accounting for the various angles at subbottom interfaces and refractions that are geometry dependent.

This paper develops a reflection coefficient inversion for geoacoustic parameters including spherical wave effects in a full-wave-field approach. Quantitative inversion results for the full wave-field forward model are compared to inversion results obtained with the plane wave approximation for a realistic simulated experiment. The full wave-field forward model uses wave number integration (OASES5) and two point ray tracing to compute the reflectivity for complex, multilayered seabed models and arbitrarily layered water columns. The geoacoustic model parameters consist of layer thickness, sound velocity, density, and attenuation for multiple (homogeneous) layers.

The inverse problem is solved using a Bayesian approach that samples the posterior probability density (PPD), incorporating both data and prior information. The solution is quantified in terms of properties of the PPD representing parameter estimates and parameter uncertainties (e.g., marginal distributions, credibility intervals), which can be computed using numerical methods for nonlinear inverse problems.8–11

The remainder of this paper is organized as follows. Section II considers the single-bounce reflectivity experiment and the data processing to yield reflection coefficients versus frequency and angle. Section III considers the Bayesian formulation of the inverse problem, data misfit functions, and methods for treating errors. Section IV develops the Bayesian reflection inversion, including data processing and averaging schemes, and the full wave-field forward model. Section V A presents results for a simulated experiment using plane-wave inversion. Section V B presents inversion results for the same data, but using the full wave-field inversion developed here. Finally, Sec. VI summarizes and discusses the results.

II. EXPERIMENT AND DATA PROCESSING

The experiment considered here (Fig. 1) involves the configuration of Holland and Osler,5 and consists of a single, fixed hydrophone r and an impulsive broadband moving source s. The source is towed near the water surface at constant speed and transmits at uniform time intervals. Seismo-
Acoustic time series (traces) are recorded for uniform increments in source–receiver range $x$. The area where the acoustic rays $\gamma_0$ interact with the seabed defines the lateral footprint of the experiment. The radius of this footprint is typically of the order of $10^2$ m, and the seabed is assumed to be laterally homogeneous over this area. Hence, the method averages over a seabed area in which no smaller lateral variations can be resolved. Because of the local experiment scale, effects of spatial and temporal variability in the water column and seabed are generally small.

For the inversions, the experiment was simulated by calculating full wave-field synthetic seismograms using OASES. The underlying geoacoustic model consists of six sediment layers over a sediment half-space (Fig. 2). Each layer is defined by layer thickness $h$, sound velocity $c$, density $\rho$, and attenuation coefficient $\alpha$. A single receiver is located at a depth of 122 m in a 150 m water column. The source was simulated at 0.35 m depth using the pulse shown in Fig. 2, transmitting at ranges of 20–560 m with a range spacing of 4.7 m. The source function was taken from experimental data by averaging the direct arrival of several traces recorded at small ranges from a boomer source and then low-pass filtering with a 3000 Hz cutoff. Green's functions for all ranges were calculated for the seabed model (Fig. 2) and then convolved with the source function. Note that the source pulse length is of the order of the layer thicknesses of the geoacoustic model, limiting the resolution of the data for seabed structure. The simulated seismoacoustic traces are shown in Fig. 3 as a function of two way travel time (TWT) and range. The direct arrival or water wave can be seen at 0.075 s and the shortest range in Fig. 3. Next, at 0.120 s, is the sediment–water interface reflection, followed by a number of subbottom reflections. The data contain no water-column multiples as the sea surface was omitted in the simulation as the source pulse included sea surface interaction for the shallow source. For the experiment geometry, water-column multiples would be well separated from the primary arrivals and would not be involved in the analysis.

The processing used to compute the reflection coefficient data from the simulated seismoacoustic traces follows Holland. The seismoacoustic traces are time windowed into one packet that contains the reflection effects of the subbottom reflectors and another packet that contains the direct wave. These packets are then processed while accounting for various experimental effects, such as source directivity, geometric spreading, refraction, absorption, and the first sea–surface interaction, to yield reflection coefficients as a function of grazing angle and frequency. In processing time series to obtain the measured reflection coefficients, the data...
are available at many frequencies and a Gaussian frequency average is applied to improve the signal to noise ratio. The data are processed for five frequency bands with center frequencies between 400 and 1600 Hz and fractional bandwidth $\kappa = 1/20$, i.e., the bandwidth is 1/20 of the center frequency. Each frequency band contains data at grazing angles of $\theta = 15–75^\circ$. The experiment results in a nonuniform angle spacing, but the data are resampled to a uniform spacing. Random Gaussian noise of standard deviation 0.05 is added to the reflection coefficient data (representative of error levels on measured reflection-coefficient data). The reflection coefficient data derived from the seismoacoustic traces are shown in Fig. 4.

III. INVERSION TECHNIQUE

A. Bayesian inversion

This section provides a brief overview of the Bayesian formulation used here for the geoacoustic inversion; more general treatments of Bayesian theory can be found elsewhere. Bayes’ rule can be written

$$P(m|d) = \frac{L(d|m)P(m)}{P(d)},$$

where $m \in \mathbb{R}^M$ and $d \in \mathbb{R}^N$ are random variables that represent the seabed model parameters and data, respectively. $P(m|d)$ is the PPD, $L(d|m)$ is the likelihood function, $P(m)$ is the model prior distribution, and $P(d)$ is the data prior distribution (a constant factor once the data are measured). The likelihood function can generally be expressed as $L(m) \propto \exp(-E(m))$ where $E(m)$ is an appropriate data error function (considered later). Equation (1) then becomes

![Reflection coefficient data](image)

**FIG. 3.** Synthetic seismoacoustic traces for the seabed model and source pulse shown in Fig. 2. The direct or water wave and several seabed reflections are visible.

![Reflection coefficient data](image)

**FIG. 4.** Reflection coefficient data (symbols) with one standard deviation error bars as derived from the seismoacoustic traces in Fig. 3. For clarity, only every third datum is shown. Solid lines are the replica data generated for the optimal seabed model from full wave-field inversion.
where the integration is over the model space \( \mathcal{M} \subset \mathbb{R}^M \) and \( \phi(m) = E(m) - \log P(m) \) is the generalized misfit. The PPD represents the full solution to the inverse problem in the Bayesian formulation. However, due to the PPD’s multidimensional nature, interpretation is non-trivial and properties such as the maximum a posteriori (MAP) model \( \hat{m} \), the mean model \( \bar{m} \), the model covariance matrix \( \mathbf{C}^{(m)} \), and marginal probability distributions \( P(m_i | d) \), must be calculated to quantify parameter estimates, uncertainties and interrelationships:

\[
\hat{m} = \text{Argmax}_P(m | d) \quad (3)
\]

\[
\bar{m} = \int_{\mathcal{M}} m' P(m' | d) \, dm' \quad (4)
\]

\[
\mathbf{C}^{(m)} = \int_{\mathcal{M}} (m' - \bar{m})(m' - \bar{m})^T P(m' | d) \, dm' \quad (5)
\]

\[
P(m_i | d) = \int_{\mathcal{M}} \delta(m'_i - m_i) P(m' | d) \, dm', \quad (6)
\]

where \( \delta \) denotes the Dirac delta function. Higher-dimensional marginal distributions can be defined similar to Eq. (6). Uncertainties of parameter estimates can also be quantified in terms of highest probability density credibility intervals. The \( \beta \% \) HPD interval is defined as the interval of minimum width that contains \( \beta \% \) of the area of the marginal probability distribution. Interrelations of model parameters can be quantified by the model correlation matrix \( R_{ij} = C_{ij} / (C_{ii} C_{jj})^{1/2} \). Although analytic solutions to Eqs. (3)–(6) exist for linear inverse problems, nonlinear problems such as geoacoustic inversion must be solved numerically.

MAP estimates, Eq. (3), can be found by numerical minimization of \( \phi(m) \), such as adaptive simplex simulated annealing (ASSA), an efficient hybrid optimization algorithm that combines the local downhill-simplex method with a very fast simulated annealing global search.\(^8\)\(^9\) The integrals of Eqs. (4)–(6) are computed here using the Markov-chain Monte Carlo method of fast Gibbs sampling\(^8\)\(^9\) (FGS) to sample \( \phi(m) \). FGS applies an adaptive Metropolis Gibbs sampling scheme\(^17\)\(^18\) in a principal-component parameter space where the coordinate axes align with the dominant correlation directions. The rotation matrix is obtained during an initial burn-in\(^19\) phase where the samples are not used for the final PPD estimate. Convergence of the burn-in phase is monitored by running independent samples and intercomparing their correlation matrices. Once a satisfactory rotation matrix is obtained, sampling of the PPD begins using a number of independent samples with convergence monitored in terms of the difference between the marginal distributions computed for each sample. The final integral estimates are based on the union of all samples collected after burn-in.

**B. Massively parallel inversion algorithms**

ASSA and FGS have been optimized on scalar computers in the past to perform inversions efficiently.\(^8\)\(^9\) However, the speed of the algorithm depends strongly on the performance of the forward model. To conduct research using computationally demanding forward models more efficiently, the ASSA and FGS codes were implemented for massively parallel computers using the message-passing interface.\(^20\) The performance gain for parallel implementations depends strongly on the granularity of the algorithm, i.e., how independently different processes in the algorithm can perform computations without the need of intercommunication. As ASSA optimization is essentially a serial algorithm, its granularity is small. The main performance gain for optimization was obtained by parallelizing the forward model by processing many independent acoustic frequencies simultaneously.

As FGS is a Monte Carlo method, it belongs to a group of algorithms that are commonly referred to as *embarrassingly parallel* and have large granularity. FGS was implemented for parallel computers by distributing the sampling of the PPD over a large number of central processing units (CPUs). For the burn-in phase, each CPU samples for a certain number of forward steps and then passes the sample to the master CPU to carry out a convergence test of the various model parameter correlation estimates. Once convergence of the rotation matrix is obtained, the actual sampling of the PPD begins in parallel on all CPUs. It was found that after the burn-in, the performance scaled almost linearly with the number of CPUs used. The algorithm was tested with up to 80 CPUs with a speedup factor of close to 80, reducing total run times from several days to less than 1 h.

**C. Likelihood function**

Formulating the likelihood function \( \mathcal{L}(m) \) requires specifying the data uncertainty distribution, including both measurement errors and theory errors. In general, the Bayesian inversion outlined earlier can be applied with arbitrary uncertainty distributions. In practice, however, the lack of specific knowledge about error distributions often suggests a mathematically simple distribution be assumed. In particular, Gaussian distributions are commonly considered, with their statistical parameters estimated from the data.

Consider reflection coefficient data as a function of grazing angle \( \theta \) (Fig. 1), given at several independent frequencies. For \( N \) observed data \( d_i \) at each of \( F \) frequencies with unbiased Gaussian-distributed random errors, the likelihood function is given by

\[
\mathcal{L}(m) \propto \prod_{i=1}^{F} \exp \left( - \frac{1}{2} (d_i - d_i(m))^T \mathbf{C}^{(d)}_{i}^{-1} (d_i - d_i(m)) \right),
\]

where \( d_i(m) \) are the replica data computed for model \( m \) and \( \mathbf{C}^{(d)}_{i} \) are the \( F \times F \) data covariance matrices.\(^11\)\(^12\)\(^13\) The data misfit function is given by the negative log likelihood.

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\[ E(m) = \sum_{i=1}^{F} \frac{1}{2} (d_i - d_i(m))^T (C_i^{(d)})^{-1} (d_i - d_i(m)). \]  

In practical applications, the covariance matrices \( C_i^{(d)} \) are not known. Depending on the data error distribution, different approaches can be applied to estimate the covariance matrices. If the data do not have significantly correlated errors (as in this simulation), the covariance matrices can be approximated as diagonal, \( C_i^{(d)} = \sigma_i I \), where \( I \) is the identity matrix and \( \sigma_i \) is the standard deviation at the \( i \)th frequency. The likelihood function then can be written

\[ \mathcal{L}(m) \propto \prod_{i=1}^{F} \exp \left( -\sum_{j=1}^{N} \frac{(d_{ij} - d_{ij}(m))^2}{2\sigma_i^2} \right), \]

where \( d_{ij} \) represents the \( j \)th datum at the \( i \)th frequency. A maximum likelihood (ML) estimate for \( \sigma_i \) can be found by maximizing \( \mathcal{L}(m) \) over \( \sigma_i \) and \( m \) to yield

\[ \hat{\sigma}_i = \left( \frac{1}{N} \sum_{j=1}^{N} (d_{ij} - d_{ij}(\hat{m}))^2 \right)^{1/2}, \]

where the ML estimate \( \hat{m} \) is found by minimizing the misfit obtained by substituting Eq. (10) into Eq. (9) to yield

\[ E(m) = \sum_{i=1}^{F} \frac{N}{2} \log \left( \sum_{j=1}^{N} (d_{ij} - d_{ij}(m))^2 \right). \]

The ML estimate \( \hat{\sigma}_i \) can then be used in sampling the PPD by applying the data misfit function

\[ E(m) = \sum_{i=1}^{F} \sum_{j=1}^{N} \frac{(d_{ij} - d_{ij}(m))^2}{2\sigma_i^2}. \]

IV. FORWARD MODELING

A. Frequency and range averaging

The full wave-field forward model applied here to compute replica data is computationally intensive. This, combined with the number of forward computations needed for a typical ASSA or FGS run (in the order of 100,000 seabed models), can make it slow to carry out inversions based on reflection coefficients at multiple frequencies, even on sophisticated supercomputers. Further, the measured data are frequency averaged to improve the signal-to-noise ratio. Due to the strong frequency dependence of the reflection coefficient, the forward model must also include similar frequency averaging for the replica data. Depending on the bandwidth and the particular environmental model, the number of frequencies required for averaging varies. In most cases, 8 or 9 frequencies have proven to provide sufficient averaging.

Figure 5 shows the reflection coefficient over range and frequency for the seabed model in Fig. 2 consisting of six layers over a semiinfinite half-space. The cross shown in Fig. 5 suggests that averaging over a certain range interval could have a similar effect to averaging over a certain band of frequencies. The advantage of averaging over range rather than frequency is that the wave number integration used to calculate the replica acoustic fields automatically provides the field at a dense range spacing. Thus, spatial averaging can be carried out at almost no extra computational cost, whereas the computational cost of frequency averaging scales with the number of frequencies used in the average.

Harrison and Harrison\textsuperscript{14} illustrated a simple relationship between frequency and range averaging for broadband sonar. Consider a Gaussian frequency average defined as

\[ I_f = \frac{\int \Psi(f, r_0) \exp(- (f - f_0)^2/(\kappa f_0)^2) df}{\int \exp(- (f - f_0)^2/(\kappa f_0)^2) df}, \]

where \( f_0 \) denotes the center frequency of the average, \( \kappa \) is the fractional bandwidth, and \( \Psi \) is the acoustic field which is explicitly given as a function of frequency and range. A range average can also be defined as

\[ I_r = \frac{\int \Psi(f_0, r) \exp(- (r - r_0)^2/(\kappa r_0)^2) dr}{\int \exp(- (r - r_0)^2/(\kappa r_0)^2) dr}, \]

where \( r_0 \) is the center range and \( \kappa \) is the same fraction as above. Equation (14) uses a sliding Gaussian window of width proportional to the center range. Harrison and Harrison\textsuperscript{14} showed for Lloyd’s mirror (i.e., a point source in a semiinfinite half-space with perfectly reflecting interface) that the averages in Eqs. (13) and (14) are identical. In other examples they illustrated that the two are very similar. Depending on the choice of \( \kappa \), the Gaussian frequency window can vary from monochromatic to broadband. The cross in Fig. 5 indicates a fractional bandwidth of \( \kappa=5 \) for a center of \( f=1000 \) Hz and \( r=250 \) m. Qualitative examination of the features along either lines suggests a very similar averaging effect for this example.

Figure 6 compares frequency and range averages for a simulated experiment. The seabed consists of six sediment layers over a sediment halfspace (Fig. 2); data were generated as time series (synthetic seismacoustic traces from
Green’s functions convolved with the source pulse in Fig. 2) and then processed to resemble reflection coefficients. The agreement is excellent for all frequencies and angles. Other simulations showed that the agreement degrades for frequencies much below those shown in Fig. 6. However, for the frequency band of interest in this work, the range average reduces computation time by an order of magnitude at no significant loss in accuracy. Further, the computational expense for lower frequencies is considerably lower, making it less important to replace frequency averaging with range averaging for low frequencies.

B. Plane wave reflection coefficient forward model

The forward models used in this paper approximate the seabed as a layered lossy fluid. This is justified, since shear velocities in fine grained sediments are low, and earlier inversion studies that treated the seabed as a halfspace showed that the reflection coefficient is relatively insensitive to shear properties and that ignored shear properties do not result in a bias for the other physical properties.

As indicated in Fig. 1, the experiment uses a point source generating spherical waves, and, therefore, the measured reflectivity contains spherical wave effects. Depending on the geometry of the experiment and the environment at the measurement site, these effects may be negligible, or may be important to take into account to recover meaningful estimates of the geoaoustic parameters.

For shallow water, low sound velocities, small penetration depth of interest, and certain frequency ranges, a plane wave assumption can be reasonable. In that case, the forward model consists of a well established recursive calculation (see Chapter 1.6 in Ref. 24), referred to here as the plane wave forward model. For cases where the plane wave assumption is not sufficient, the following sections develop a full wave-field forward model.

C. Reflection of spherical waves from interfaces

For finite distance between source and interface or receiver (i.e., distances less than many wavelengths), full wave-field (spherical wave) effects must be taken into account to completely describe the reflection problem. The classic work on reflection of spherical waves is by Sommerfeld; this section follows the derivations in Brekhovskikh and Godin. For two homogeneous media, the total acoustic field of a spherical wave at a point that includes both direct and reflected components can be written as

$$\Psi = \frac{\exp(iKR)}{R} + \Psi_{\text{ref}},$$

(15)

where $\Psi_{\text{ref}}$ is the reflected wave, $k$ is the wave number, and $R$ is the distance from the source. $\Psi_{\text{ref}}$ can be written as a superposition of reflected plane waves

$$\Psi_{\text{ref}} = \frac{ik}{2\pi} \int_{\pi/2}^{\pi} \int_0^{2\pi} \exp(ik(x \cos \theta \cos \phi + y \cos \theta \sin \phi + (z + z_0)\sin \theta)) \psi(\theta) \cos \theta d\theta d\phi,$$

(16)

where $x$, $y$, and $z$ are spatial coordinates, $z + z_0$ is the source-receiver vertical separation with respect to the interface, $\phi$ is
the azimuth, and $V(\theta)$ is the plane wave reflection coefficient.

It can be shown that the integral over $\phi$ in Eq. (16) reduces to a Bessel function of zeroth order. By rearranging the integral limits in Eq. (16), the Bessel function can be expressed in terms of Hankel functions

$$
\Psi_{ref} = \frac{ik}{2} \int_{-\pi/2-i\varepsilon}^{\pi/2-i\varepsilon} H_0^{(1)}(u) \times \exp(ik(z + z_0)\sin \theta)V(\theta)\cos \theta \, d\theta,
$$

where $u=kr \cos \theta$. Equation (17) can be used to calculate the up-going spherical wave and thus the up-going energy (including reflected and lateral waves) for an arbitrary set of layers, given the plane wave reflection coefficient $V(\theta)$. The spherical reflection coefficient $V_s$ can then be defined as to include all effects but spherical spreading

$$
\Psi_{ref} = \frac{\exp(ikR_s)}{R_1}
$$

$$
= \frac{ik}{2} \int_{-\pi/2-i\varepsilon}^{\pi/2-i\varepsilon} H_0^{(1)}(u) \exp(ik(z + z_0)\sin \theta)V(\theta)\cos \theta \, d\theta.
$$

(18)

A method to yield $V_s$ [Eq. (18)], based on calculating the total acoustic field is discussed in Sec. IV D.

For finite frequencies, the reflection is not from a single point but rather from a finite volume (a Fresnel volume) around the specular point. In essence, a hydrophone registers energy from a range of angles for a planar reflector. Spherical wave effects are important when the Fresnel zone spans an angular range over which the reflection coefficient changes significantly. For instance, the reflection coefficient can change rapidly with angle, particularly around the critical angle. The steepest descent solution for the reflected wave is given by

$$
\Psi_{ref} = \frac{\exp(ikR)}{R} \left( \frac{V(\theta_s) - i}{2kR} (V''(\theta_s) + V'(\theta_s)\tan(\theta_s)) \right),
$$

(19)

where $V'$ and $V''$ are the first and second derivative of the plane wave reflection coefficient with respect to angle and $\theta_s$ is the specular angle. The dependence of the reflected field on the derivatives of the reflection coefficient with respect to angle illustrates the importance of full wave-field effects when the reflection coefficient changes rapidly with angle. Hence, a plane wave solution is expected to work well when the total sediment thickness is much smaller than the water depth and when the reflection coefficient changes slowly with angle, but may break down for increased depth of interest and cases involving rapid changes with angle.

The breakdown of the plane-wave approximation for plane reflectors at large depth below the water sediment interface can also be illustrated using ray theory. In the plane wave case, every sediment interface contributes to the reflection coefficient at the same angle. However, at finite distance from the source, as illustrated in Fig. 7, the energy for each sediment interface originates at a different angle. This effect becomes more significant with larger reflection depth and means that spherical wave effects are more important for deeper layers in an environmental model.

To illustrate the above points, Fig. 8 shows a simulation of plane wave and spherical wave reflections for the case of a single interface between two halfspaces. The simulation was carried out at a frequency of 500 Hz for two different environmental models that differ only in the sound velocity of the lower halfspace. The source-receiver vertical separation was 178 m and one seabed model was given by $c_w =1511 \text{ m/s}$ and $\rho_w =1.029 \text{ g/cm}^3$, $\alpha_w =0 \text{ dB/\lambda}$, $c_1 =1700 \text{ m/s}$, $\rho_1 =1.4 \text{ g/cm}^3$, and $\alpha_1 =0.06 \text{ dB/\lambda}$. For the second seabed model, $c_1$ was set to 1670 m/s. It can be seen that the greatest differences occur near the critical angle (25°) and that $V_s$ is consistently lower than $V$ around the critical angle. As Stronger has shown, the amount of displacement of the critical angle depends on the source receiver height above the interface as well as on the frequency. With increasing frequency or source-receiver vertical separation, the displacement of the critical angle decreases. Considering that the Fresnel zone is not symmetric around the specular point, this results in an effective critical angle that is shifted towards lower angles. The shift toward lower angles will generally

![Figure 7](image_url7.png)

**FIG. 7.** Reflection of a spherical wave (represented by rays $\gamma_1$, $\gamma_2$, and $\gamma_3$) from multiple interfaces (represented by three layers with sound velocities $c_i$, densities $\rho_i$, and attenuations $\alpha_i$); the point $r$ registers energy contributions from different angles $\theta_i$.

![Figure 8](image_url8.png)

**FIG. 8.** Comparison of plane wave ($V$) and spherical wave ($V_s$) reflection coefficients (solid and dashed lines respectively) for an interface between two halfspaces at 500 Hz. Two different sets of parameters are considered (see the text). They only differ in the sound speed specified for the lower halfspace, 1670 m/s (thin lines) and 1700 m/s (thick lines).
result in a negative bias of sound velocities in inversions carried under the plane wave assumption. The phenomenon of reflection coefficient values greater than one, evident in Fig. 8, results from lateral waves.

For a rigorous examination of the reflection phenomena, the intuitive effects described above are not sufficient, and contributions from complex angles or inhomogeneous waves must be considered as well. All of these effects are included in the spherical wave reflection coefficient forward model described in the following section.

D. Spherical wave reflectivity forward model

For reflection coefficients that change rapidly with angle and for complex environments (many layers, many wavelengths depth below the seafloor), spherical wave effects are often significant. Harrison and Nielsen27,28 showed that plane wave and spherical wave reflection coefficients can be significantly different even for large source-receiver vertical separation (e.g., 210 m source-receiver vertical separation at 500 Hz). In this paper, a forward model is developed that derives a reflection coefficient, including spherical wave effects, from complex acoustic fields. The full acoustic fields are computed by wavenumber integration20 and then processed using ray tracing to resemble reflection coefficients that take full spherical wave effects into account. Several numerical optimization techniques are applied to run the computationally intensive forward model as efficiently as possible.

Calculation of the full acoustic field $\Psi$ is based on wave number integration ($\text{OASES}^7$). For single-bounce reflection problems, the field can be split into two parts, a direct wave and a seabed response term

$$
\Psi = \frac{\exp(i \int_{R_1} k(u) du)}{R_1} + \frac{\exp(i \int_{R_2} k(u) du)}{R_2}.
$$

(20)

where $k$ is the wave number and $R_1$ and $R_2$ are the path length of the direct and bottom bounce paths, respectively. $V_s$ is a measure for the seabed response which can be recovered by rearranging Eq. (20) to yield

$$
V_s = \frac{\Psi - \exp(i \int_{R_1} k(u) du) / R_1}{\exp(i \int_{R_2} k(u) du) / R_2}.
$$

(21)

The path lengths $R_1$ and $R_2$ and wave number integrals along the rays are determined using ray tracing in the water column, employing a depth dependent sound velocity profile. Hence, $V_s$ quantifies the seabed response by correcting the field $\Psi$ for spreading loss of direct and bottom reflected paths. All spherical wave effects (including possible lateral and inhomogeneous waves) are contained in $V_s$ and will be accounted for in the inversion. It should be noted that Eq. (20) is a convenient way to process the acoustic field replica data to resemble the processing of the measured data, and is therefore suitable for the inversion. For convenience, $V_s$ is referred to as the spherical wave reflection coefficient, although more than just reflection effects enter the coefficient.

As reflection coefficients show a strong frequency dependence, frequency averaging must be applied to be consistent with the measured data. Here, the frequency average is replaced by an equivalent range average, as outlined in Sec. IV A. With the shortest range being in the order of 10$^1$ m, the wave number integration model is run with full Hankel transforms, eliminating any far field approximation. This ensures precise modeling of the acoustic field even at small ranges, at the price of higher computational effort.

This forward model is powerful and general and can calculate the spherical wave reflection coefficient for arbitrary sound velocity profiles in the water and arbitrary layering in the seabed. It can also account for interface roughness, and shear and gradient layers can be built into the seabed model parameterization, although these features are not used here.

V. INVERSION RESULTS

A. Plane wave inversion

To compare the performance of the plane wave forward model to the full wave-field forward model, both forward models are applied in inversion. First, the Bayesian geoaoustic inversion with the plane wave forward model is applied to the reflectivity data extracted from the synthetic seismograms (Figs. 3 and 4). The prior bounds, given in Table I, are chosen to be representative of the information that would

![FIG. 9. Fit of the MAP replica data (solid line) to the simulated experimental data (dotted line) for plane-wave inversion. The dashed line shows data generated with the plane wave forward model for the true parameters.](image-url)
typically be available, and are wide enough so that the data rather than the prior primarily determine the inversion results. The data are first inverted using ASSA under the assumption of unknown error magnitudes at each frequency to minimize misfit function Eq. (11). The resulting ML seabed model was used to find a ML estimate of the error standard deviation at each frequency according to Eq. (10). These standard deviations were then used in Eq. (12) and FGS was applied to sample the PPD as a function of the 27 unknown parameters. Figure 9 shows that the data fit of the MAP seabed model is reasonably good for clarity, two out of five frequencies are shown; the results are representative of all frequencies. Figure 9 also shows data that were generated with the plane wave forward model using the true geoaoustic parameters. There is a noticeable difference between the simulated experiment data and plane wave data for the true seabed model, especially at low frequencies.

Figure 10 shows the plane-wave inversion results in terms of the mean seabed model and 95% HPD credibility intervals (see Sec. III A). It can be seen that, although velocities and densities of the upper three layers are reasonably well determined, deeper layers cannot be recovered with the plane wave forward model. The velocity profile shows strong negative biases at depth, as expected from the analysis in Sec. IV D, and does not represent the true profile well. Biases also occur for the density profile; however, the 95% HPD credibility intervals are wide enough to include the true seabed model with the exception of the deepest layer. Further, some layer thicknesses show significant discrepancies from the true seabed model and the 95% HPD credibility intervals do not include the true seabed model (see also Table II). The credibility intervals for the attenuation profile generally extend over much of the search bounds (Table I), indicating little ability to resolve this parameter. Numerical values for the inversion results are given in Table II.

The inversion results in Fig. 10 show that the plane wave forward model is not sufficient to recover the complex structure in this case. Even though the data can be fit reasonably well, this fit is obtained by introducing a substantial bias in the parameter estimates to account for the theory error. Not only is the data fit misleading about the quality of the results, but the HPD credibility intervals also provide false confidence. In this case, a full wave-field forward model must be considered.

**B. Spherical wave inversion**

This section applies the Bayesian inversion using the full wave-field forward model to the reflection data derived

<table>
<thead>
<tr>
<th>Layer</th>
<th>h (m)</th>
<th>c (m/s)</th>
<th>( \rho ) (g/cm(^3))</th>
<th>( \alpha ) (dB/( \lambda ))</th>
</tr>
</thead>
<tbody>
<tr>
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<td>True</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
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<td>1.44–1.53</td>
<td>1540</td>
<td>1523–1538</td>
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<tr>
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<td>3.0</td>
<td>3.05–3.16</td>
<td>1520</td>
<td>1521–1534</td>
</tr>
<tr>
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<td>1.0</td>
<td>0.90–1.02</td>
<td>1560</td>
<td>1529–1548</td>
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<tr>
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<td>7.03–7.07</td>
<td>1600</td>
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<td>1600–1607</td>
</tr>
<tr>
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<td>1650</td>
<td>1603–1607</td>
<td>2.20</td>
<td>1.86–2.14</td>
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</table>

**FIG. 10.** Bayesian inversion results using the plane wave forward model. The solid line indicates the a posteriori mean seabed model; shaded areas are the 95% HPD credibility intervals. The dashed line represents the true seabed model.
from seismo-acoustic traces (Figs. 3 and 4). As before, ASSA optimization is used to obtain a maximum likelihood estimate of the error standard deviations at each frequency, which are then used in FGS to sample the PPD. The fit to the data is shown in Fig. 4. Figure 11 shows the sound-velocity, density, and attenuation profiles and their associated uncertainties in terms of 95% HPD credibility intervals. The \textit{a posteriori} mean seabed model is very close to the true seabed model for both the velocity and density profiles. Layer thicknesses are also matched well. The velocity tends to be resolved better (smaller uncertainty) in thicker layers. In general, the uncertainties in the parameter estimates grow with depth, particularly for the density profile. Attenuation is not resolved particularly well, with credibility intervals covering much of the prior bounds given in Table I. The best resolution for attenuation was obtained in the thickest layers (see Table III). Comparing the full wave-field results in Fig. 11 to the plane wave approximation results in Fig. 10 indicates that a substantial improvement is obtained using the full wave-field forward model.

VI. DISCUSSION AND SUMMARY

This paper developed a full wave-field Bayesian inversion for reflection coefficient data derived from a single bounce experiment. The inversion was applied to data from a simulated experiment, and results were compared to those obtained with an inversion using the standard plane wave approximation. The full wave-field inversion algorithm is based on a forward model that use complex acoustic fields, obtained through wavenumber integration, to calculate the replica reflection coefficients. Hence, the replica reflection coefficients contain full wave-field effects which are not included in a plane wave approximation. Several techniques were applied to carry out the computational intensive inversions efficiently, including replacing frequency with range averaging and parallel processing.

The data considered here were generated by computing synthetic seismograms for a seabed model including six layers and a halfspace. Green’s functions were computed for the seabed model and convolved with a source pulse from measured data. The synthetic seismograms were then processed to yield reflection coefficients as a function of angle and frequency, with Gaussian noise added. Both inversions (plane wave and full wave-field forward models) were applied to this data set. The presented results depend on experiment geometry and full wave-field effects increase with depth of a layer beneath the sea floor.

![Fig. 11. Bayesian inversion results for the spherical wave forward model. The solid line indicates the a posteriori mean seabed model; shaded areas are the 95% HPD intervals. The dashed line represents the true seabed model. The inversion results and their associated uncertainties agree well with the true seabed model.](image)

<table>
<thead>
<tr>
<th>Layer</th>
<th>$h$ (m)</th>
<th>$c$ (m/s)</th>
<th>$\rho$ (g/cm$^3$)</th>
<th>$\alpha$ (dB/\lambda)</th>
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<tr>
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</table>

TABLE III. Numerical values for true parameters and inversion results (True credibility intervals) for the spherical wave inversion.
Comparing the inversion results for the two forward models indicated the plane wave assumption was insufficient for this case. The environmental model could not be resolved at depth using the plane wave forward model, resulting in biased estimates and misleading credibility intervals. The full wave-field inversion provided substantial improvements over the plane wave inversion. In particular, the sediment velocity profile was estimated to high accuracy, with smaller uncertainties associated with thicker layers. Density was also well resolved with reasonable credibility intervals. Uncertainties for velocity and density generally increase with depth although smaller velocity uncertainties are associated with thicker layers.

ACKNOWLEDGMENTS

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23M. D. Richardson, Shallow Water Acoustics (China Ocean Press, Beijing, 1997).